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Special Elements of a Ternary Semiring

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ABSTRACT
In this paper we study the notion of some special elements such as identity, zero, absorbing, additive idempotent, idempotent, multiplicatively sub-idempotent, regular, Intra regular, completely regular, *g*–regular, invertible and the ternary semirings such as zero sum free ternary semiring, zero ternary semiring, zero divisor free ternary semiring, ternary semi-integral domain, semi-subtractive ternary semiring, multiplicative cancellative ternary semiring, Viterbi ternary semiring, regular ternary semiring, completely ternary semiring and characterize these ternary semirings.
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***Key Words* :** Multiplicatively sub-idempotent, regular, Intra regular, completely regular, *g*–regular, zero sum free ternary semiring, zero ternary semiring, zero divisor free ternary semiring, ternary semi-integral domain, semi-subtractive ternary semiring, Viterbi ternary semiring,

I. INTRODUCTION

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces and the like. This provides sufficient motivation to researchers to review various concepts and results.

The theory of ternary algebraic systems was studied by LEHMER [9] in 1932, but earlier such structures were investigated and studied by PRUFER in 1924, BAER in 1929.

Generalizing the notion of ternary ring introduced by Lister [10], Dutta and Kar [6]
introduced the notion of ternary semiring. Ternary semiring arises naturally as follows, consider the ring of integers *Z* which plays a vital role in the theory of ring. The subset *Z*+ of all positive
integers of *Z* is an additive semigroup which is closed under the ring product, i.e. *Z*+is a semiring. Now, if we consider the subset *Z*− of all negative integers of *Z*, then we see that *Z*− is an additive semigroup which is closed under the triple ring product (however, *Z*− is not closed under the binary ring product), i.e. *Z*− forms a ternary semiring. Thus, we see that in the ring of integers *Z*, *Z*+ forms a semiring whereas *Z*− forms a ternary semiring.

II. PRELIMINARIES :

DEFINITION 1.1 : A nonempty set T together with a binary operation called addition and a ternary multiplication denoted by [ ] is said to be a ***ternary semiring*** if T is an additive commutative semigroup satisfying the following conditions :
i) $[[abc]de] = [a[bcd]e] = [ab[cde]]$,
ii) $[(a + b)cd] = [acd] + [bcd]$,
iii) $[a(b + c)d] = [abd] + [acd]$,
iv) $[ab(c + d)] = [abc] + [abd]$ for all $a; b; c; d; e \in$ T.
Throughout T will denote a ternary semiring unless otherwise stated.

NOTE 1.2 : For the convenience we write $x_1x_2x_3$ instead of $[x_1x_2x_3]$

NOTE 1.3 : Let T be a ternary semiring. If A,B and C are three subsets of T , we shall denote the set ABC = $\{\Sigma abc : a \in A, b \in B, c \in C\}$.

NOTE 1.4 : Let T be a ternary semiring. If A,B are two subsets of T , we shall denote the set A + B = $\{a + b : a \in A, b \in B\}$.

NOTE 1.5 : Any semiring can be reduced to a ternary semiring.

EXAMPLE 1.6 : Let T be an semigroup of all $m \times n$ matrices over the set of all non negative rational numbers. Then T is a ternary semiring with matrix multiplication as the ternary operation.

EXAMPLE 1.7 : Let $S = \{. . . ,-2i, -i, 0, i, 2i, . . . \}$ be a ternary semiring withrespect to addition and complex triple multiplication.

EXAMPLE 1.8 : The set T consisting of a single element 0 with binary operation defined by 0 + 0 = 0 and ternary operation defined by 0.0.0 = 0 is a ternary semiring. This ternary semiring is called the ***null ternary semiring*** or the ***zero ternary semiring.***

EXAMPLE 1.9 : The set T = { 0, 1, 2, 3, 4 } is a ternary semiring with respect to addition modulo 5 and multiplication modulo 5 as ternary operation is defined as follows :

$+_5$	0	1	2	3	4
0	0	1	2	3	5
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\times_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

EXAMPLE 1.10 : The set C of all real valued continuous functions defined in the closed interval [0, 1] is a ternary semiring with respect to the addition and ternary multiplication of functions defined as follows :

$(f + g)(x) = f(x) + g(x)$ and $(fgh)(x) = f(x)g(x)h(x)$, where f, g, h are any three members of C.

DEFINITION 1.11 : A ternary semiring T is said to be ***commutative ternary semiring*** provided $abc = bca = cab = bac = cba = acb$ for all $a,b,c \in$ T.

DEFINITION 1.12 : A ternary semiring T is said to be ***quasi commutative*** provided for each $a, b, c \in$ T, there exists a odd natural number n such that

$abc = b^n ac = bca = c^n ba = cab = a^n cb$.

THEOREM 1.13 : If T is a commutative ternary semiring then T is a quasi commutative ternary semiring.

DEFINITION 1.14 : A ternary semiring T is said to be ***normal*** provided abT = Tab $\forall a,b \in$ T.

THEOREM 1.15 : If T is a quasi commutative ternary semiring then T is a normal ternary semiring.

DEFINITION 1.16 : A ternary semiring T is said to be ***left pseudo commutative*** provided $abcde = bcade = cabde = bacde = cbade = acbde$ $\forall a,b,c,d,e \in$ T.

DEFINITION 1.17 : A ternary semiring T is said to be a ***lateral pseudo commutative*** ternary semiring provide $abcde = acdbe = adbce = acbde = adcbe = abdce$ for all $a,b,c,d,e \in$ T.

DEFINITION 1.18 : A ternary semiring T is said to be ***right pseudo commutative*** provided $abcde = abdec = abecd = abdce = abedc = abced$ $\forall$ $a,b,c,d,e \in$ T.

DEFINITION 1.19 : A ternary semiring T is said to be ***pseudo commutative***, provided T is a left pseudo commutative, right pseudo commutative and lateral pseudo commutative ternary semiring.

III. SPECIAL ELEMENTS IN A TERNARY SEMIRING:

DEFINITION 2.1 : An element a in a ternary semiring T is said to be an additive identity provided $a + x = x + a = x$ for every $x \in$ T.

DEFINITION 2.2 : An element a of ternary semiring T is said to be ***left identity*** of T provided $aat = t$ for all $t \in$ T.

NOTE 2.3 : Left identity element a of a ternary semiring T is also called as ***left unital element.***

DEFINITION 2.4 : An element a of a ternary semiring T is said to be a ***lateral identity*** of T provided $ata = t$ for all $t \in$ T.

NOTE 2.5 : Lateral identity element a of a ternary semiring T is also called as ***lateral unital element.***

DEFINITION 2.6: An element a of a ternary semiring T is said to be a ***right identity*** of T provided $taa = t$ $\forall t \in$ T.

NOTE 2.7 : Right identity element a of a ternary semiring T is also called as ***right unital element.***

DEFINITION 2.8 : An element a of a ternary semiring T is said to be a ***two sided identity*** of T provided $aat = taa = t$ $\forall$ $t \in$ T.

NOTE 2.9 : Two-sided identity element of a ternary semiring T is also called as ***bi-unital element***.

EXAMPLE 2.10 : In the ternary semiring Z_0^-, the element (-1) is a bi-unital element.

DEFINITION 2.11 : An element *a* of a ternary semiring T is said to be an ***identity*** provided $aat = taa = ata = t \; \forall \; t \in T$.

NOTE 2.12 : An identity element of a ternary semiring T is also called as ***unital element.***

NOTE 2.13 : An element *a* of a ternary semiring T is an ***identity*** of T iff *a* is left identity , lateral identity and right identity of T.

EXAMPLE 2.14: Let Z_0^- be the set of all non-positive integers. Then with the usual addition and ternary multiplication, Z_0^- forms a ternary semiring with identity element -1.

EXAMPLE 2.15: Let $M_{2\times 2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} / \, a,b,c,d \in z^0 \right\}$. Then ($M_{2\times 2}$, +, .) is a ternary semiring under the addition and matrix ternary multiplication. $M_{2\times 2}$ is a non-commutative ternary semiring with identity element $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and is of infinite order.

EXAMPLE 2.16: $nZ^0 = \{ 0 , n , 2n , \ldots\ldots \}$ is a ternary subsemiring of Z^0

***Proof* :** Let *kn* , *mn ln* be three elements in nZ^0, then $kn + mn = (k + m)n \in nZ^0$.
$(kn)(mn)(ln) = (klmn)n \in nZ^0$. Hence, nZ^0 is a ternary subsemiring of Z^0 without identity.

NOTE 2.17 : The identity (if exists) of a ternary semiring is usually denoted by e.

NOTATION 2.18 : Let T be a ternary semiring. If T has an identity, let T^e = T and if T does not have an identity , let T^e be the ternary semigroup T with an identity adjoined usually denoted by the symbol *e*.

In the following we introducing left zero, lateral zero, right zero, two sided zero and zero of ternary semiring.

DEFINITION 2.19: An element *a* in a ternary simiring T is said to be an additive zero provided $a + x = x + a = x$ for all $x \in T$.

DEFINITION 2.20 : An element *a* of a ternary semiring T is said to be a ***left zero*** of T provided $a + x = x$ and $abc = a \; \forall \; b, c, x \in T$.

DEFINITION 2.21 : An element *a* of a ternary semiring T is said to be a ***lateral zero*** of T provided $a + x = x$ and $bac = a \; \forall \; b, c, x \in T$.

DEFINITION 2.22 : An element *a* of a ternary semiring T is said to be a ***right zero*** of T provided $a + x = x$ and $bca = a \; \forall \; b, c, x \in T$.

DEFINITION 2.23 : An element *a* of a ternary semiring T is said to be a ***two sided zero*** of T provided $a + x = x$ and $abc = bca = a \; \forall \; b, c, x \in T$.

NOTE 2.24 : If *a* is a two sided zero of a ternary semiring T, then *a* is both left zero and right zero of T.

DEFINITION 2.25 : An element *a* of a ternary semiring T is said to be ***zero*** of T provided $a + x = x$ and $abc = bac = bca = a \; \forall \; b, c, x \in T$.

EXAMPLE 2.26 : Let Z_0^- be the set of all negative integers with zero. Then with the usual binary addition and ternary multiplication, Z_0^- forms a ternary semiring with zero.

NOTE 2.27 : If *a* is a zero of T, then a is a left zero, lateral zero and right zero of T.

NOTE 2.28: If *a* is a zero element of ternary semiring T. Then *a* is also called an absorbing element of T.

THEOREM 2.29 : If *a* is a left zero, *b* is a lateral zero and *c* is a right zero of a ternary semiring T , then *a* = *b* = *c*.

Proof : Since *a* is a left zero of T, $a + x = x$, $abc = a$ for all $a, b, c, x \in T$.
Since *b* is a lateral zero of T, $b + x = x$, $abc = b$ for all $a, b, c, x \in T$.

Since c is a right zero of T, $x + c = x$, $abc = c$ for all $a, b, c, x \in$ T.
Therefore $a + x = b + x = c + x = x$ and $abc = a = b = c$ and hence $a = b = c$.

THEOREM 2.30 : Any ternary semiring has at most one zero element.

Proof : Let a,b,c be three zeros of a ternary semiring T. Now a can be considered as a left zero , b can be considered as a lateral zero and c can be considered as a right zero of T. By theorem 1.4.30, $a = b = c$. Then T has at most one zero.

NOTE 2.31 : The zero (if exists) of a ternary semiring is usually denoted by 0.

NOTATION 2.32 : Let T be a ternary semiring. if T has a zero, let T^0 = T and if T does not have a zero , let T^0 be the ternary semigroup T with zero adjoined usually denoted by the symbol 0.

DEFINITION 2.33 : A ternary semiring T is said to be a ***strict ternary semiring*** or ***zero sum free*** provided $a + b = 0$ implies $a = 0$ and $b = 0$.

EXAMPLE 2.34 : Let Z^0 be the set of positive integers. Then Z^0 is a strict ternary semiring.

DEFINITION 2.35: An element a in a ternary semiring T is said to be an ***absorbing*** w.r.t addition provided $a + x = x + a = a$ for all $x \in$ T.

DEFINITION 2.36 : A ternary semiring in which every element is a left zero is called a ***left zero ternary semiring***.

DEFINITION 2.37 : A ternary semiring in which every element is a lateral zero is called a ***lateral zero ternary semiring.***

DEFINITION 2.38 : A ternary semiring in which every element is a right zero is called a ***right zero ternary semiring***.

DEFINITION 2.39 : A ternary semiring with 0 in which the product of any three elements equal to 0 is called ***a zero ternary semiring*** (or) ***null ternary semiring***.

DEFINITION 2.40 : A ternary semiring T is said to be ***zero divisor free*** (ZDF) if for $a, b, c \in T$, $[abc] = 0$ implies that $a = 0$ or $b = 0$ or $c = 0$.

DEFINITION 2.41 : A commutative ternary semiring (ring) is called a ***ternary semi-integral*** **(*integral,*** resp.) ***domain*** if it is zero divisor free.

DEFINITION 2.42 : A ternary semiring T is said to be ***semi-subtractive*** if for any elements $a; b \in$ T; there is always some $x \in$ T or some $y \in$ T such that $a + y = b$ or $b + x = a$.

NOTE 2.43: Each ternary ring is a semi-subtractive ternary semiring.

DEFINITION 2.44 : A ternary semiring T is said to be ***multiplicatively left cancellative*** (MLC) if $abx = aby$ implies that x = y for all $a, b, x, y \in$ T.

DEFINITION 2.45: A ternary semiring T is said to be ***multiplicatively laterally cancellative*** (MLLC) if $axb = ayb$ implies that x = y for all $a, b, x, y \in$ T.

DEFINITION 2.46 : A ternary semiring T is said to be ***multiplicatively right cancellative*** (MRC) if $xab = yab$ implies that x = y for all $a, b, x, y \in$ T.

DEFINITION 2.47 : A ternary semiring T is said to be ***multiplicatively cancellative*** (MC) if it is multiplicative left cancellative (MLC), multiplicative right cancellative (MRC) and multiplicative laterally cancellative (MLLC).

THEOREM 2.48: An multiplicative cancellative ternary semiring T is zero divisor free.

Proof: Let T be an multiplicative cancellative ternary semiring and $abc = 0$ for a; b; $c \in$ T.
Suppose $b \neq 0$ and $c \neq 0$. Then by right cancellativity, $abc = 0 = 0bc$ implies that $a = 0$. Similarly, we can show that $b = 0$ if $a \neq 0$ and $c \neq 0$ or $c = 0$ if $a \neq 0$ and $b \neq 0$. Consequently, T is zero divisor free.

For the converse part we have the following result:

THEOREM 2.49: A zero divisor free (ZDF) ternary semiring T is multiplicative cancellative (MC) whenever it is additive cancellative (AC) and semi-subtractive.

Proof: Let T be a ZDF, AC and semi-subtractive ternary semiring. Let $a; b \in$ T \ 0 be such that $abx = aby$ for $x; y \in$ T. Since T is semi-subtractive, for $x; y \in$ T there is always some $c \in$ T or some $d \in$ T such that $y + c = x$ or $x + d = y$. Let $y + c = x$. Then $aby + abc = abx$ implies $abc = 0$ (by AC) which again implies that c = 0 (since S is ZDF). Similarly, we can show that $d = 0$ when $x + d = y$. Consequently, we have $x = y$ and hence T is multiplicatively left cancelletive (MLC, for short). Similarly, it can be proved that T is multiplicatively right cancelletive (MRC) and multiplicatively leterally cancelletive (MLLC). Thus T is MC.

In the following we are introducing the notion of idempotent element of a ternary semiring.

DEFINITION 2.50 : An element a of a ternary semiring T is said to be ***additive idempotent*** element provided $a + a = a$.

DEFINITION 2.51 : An element a of a ternary semiring T is said to be an ***idempotent*** element provided $a^3 = a$.

EXAMPLE 2.52 : Let Z_6 = {0 , 1 , 2 , 3 , 4 , 5 } be the commutative ternary semiring under addition and multiplication module 6 as ternary compositions. Since $4 \in Z_6$ is an idempotent , since $4 \,.\, 4 \,.\, 4 = 4(mod 6)$.

NOTE 2.53 : The set of all idempotent elements in a ternary semiring T is denoted by E(T).

NOTE 2.54 : Every identity, zero elements are idempotent elements.

In the following we introduce proper idempotent element and idempotent ternary semiring.

DEFINITION 2.55 : An element a of a ternary semiring T is said to be a ***proper idempotent*** element provided a is an idempotent which is not the identity of T if identity exists.

DEFINITION 2.56 : A ternary semiring T is said to be an ***idempotent ternary semiring*** or ***ternary band*** provided every element of T is an idempotent.

DEFINITION 2.57 : An element a of a ternary semiring T is said to be ***multiplicatively sub-idempotent*** provided $a + a^3 = a$.

DEFINITION 2.58 : A ternary semiring T is said to be a ***sub-idempotent ternary semiring*** provided each of its element is sub-idempotent.

DEFINITION 2.59 : A ternary semiring T is said to be ***Viterbi ternary semiring*** provided T is additively idempotent and multiplicatively sub-idempotent.

NOTE 2.60 : A Viterbi ternary semiring is a ternary semiring if $a + a = a$ and $a + a^3 = a$ for all $a \in$ T.

NOTE 2.61 : The concept of Viterbi ternary semiring is taken from the book of Jonathan S. Golan [7], entitled "Semirings and Affine Equations over them: Theory and Applications".

THEOREM 2.62 : Every ternary semiring T in which T is left singular semigroup w.r.t addition and left singular ternary semigroup w.r.t ternary multiplication, then T is a Viterbi ternary semiring.

***Proof* :** By the hypothesis T is left singular semigroup w.r.t addition and left singular ternary semigroup w.r.t ternary multiplication. i.e., $a + b = a$ and $ab^2 = a \;\forall\, a, b \in$ T.

Consider $a + b = a$. Taking $a = b \Rightarrow a + a = a \rightarrow (1)$ and

$ab^2 = a$. Taking $a = b \Rightarrow a.\, a^2 = a \Rightarrow a^3 = a \Rightarrow a + a^3 = a + a = a \Rightarrow a + a^3 = a \longrightarrow (2)$

Therefore from (1) and (2), T is a Viterbi ternary semiring.

NOTE 2.63 : The converse of the theorem 1.4.62, is not necessarily true. This is evident from the following example.

EXAMPLE 2.64 :

+	a	b
a	a	a
b	a	b

.	a	b
a	a	a
b	b	b

THEOREM 2.65: Let T be a ternary semiring satisfying the identity $a + aba + a = a$ for all $a, b \in$ T. If T contains the ternary multiplicative identity which is also an additive identity, then T is a multiplicatively sub-idempotent ternary semiring.

***Proof* :** Since $a + aba + a = a$ for all $a, b \in$ T. Let e be the multiplicative identity in T is also an additive identity. i.e., $aee = eae = eea = a$ and $a + e = e + a = a$. Given $a + aba + a = a$ for all $a, b \in$ T. Taking $a = b$. Then $a + a^3 + a = a \Rightarrow a + a(a^2 + e) = a \Rightarrow a + a^3 = a + 0$ $\Rightarrow a + a^3 = a$ for all $a \in$ T. Therefore T is multiplicatively sub-idempotent ternary semiring.

THEOREM 2.66 : Let T be a multiplicatively idempotent ternary semiring. If T satisfying the identity $a + aba + a = a$ for all $a, b \in$ T. Then T is additively idempotent.

***Proof* :** Let $a \in$ T. Consider $a + a = (a + a)^3$ for all $a \in$ T.

$a + a = (a + a)^3 = (a + a)(a + a)(a + a) = a^3 + a^3 + a^3 + a^3 + a^3 + a^3 + a^3 + a^3$

$= (a + a^3 + a) + (a + a^3 + a) + a + a = a + a + a + a = a + a^3 + a^3 + a$

$= a + a(a + a)a + a = a$. Therefore T is additively idempotent.

In the following we are introducing regular element and regular ternary semiring.

DEFINITION 2.67 : An element a of a ternary semiring. T is said to be ***regular*** if there exist $x, y \in$ T such that $axaya = a$.

DEFINITION 2.68 : A ternary semigroup T is said to be ***regular ternary semiring*** provided every element is regular.

THEOREM 2.69 : Every idempotent element in a ternary semiring is regular.

***Proof* :** Let a be an idempotent element in a ternary semiring T.

Then $a = a^3 = a.a^2 = a^3.a^2 = a.a.a.a.a$. Therefore a is regular element.

In the following we are introducing the notion of left regular, lateral regular right regular, intra regular and completely regular elements of a ternary semiring and completely regular ternary semiring.

DEFINITION 2.70 : An element a of a ternary semiring T is said to be ***left regular*** if there exist $x, y \in$ T such that $a = a^3xy$.

DEFINITION 2.71 : An element a of a ternary semiring T is said to be ***lateral regular*** if there exist $x, y \in$ T such that $a = xa^3y$.

DEFINITION 2.72 : An element a of a ternary semiring T is said to be ***right regular*** if there exist $x, y \in$ T such that $a = xya^3$.

DEFINITION 2.73 : An element a of a ternary semiring T is said to be ***intra regular*** if there exist $x, y \in$ T such that $a = xa^5y$.

DEFINITION 2.74 : An element a of a ternary semring T is said to be ***completely regular*** if there exist $x, y \in$ T such that

(i) $axaya = a$
(ii) $axa = xaa = aax = aya = yaa = aay = axy = yxa = xay = yax$.
(iii) $a = a + x + a$
(iv) $[aa(a + x)] = a + x$.

DEFINITION 2.75 : A ternary semigroup T is said to be a ***completely regular ternary semiring*** provided every element in T is completely regular.

THEOREM 2.76 : Let T be a ternary semiring and $a \in$ T. If a is a completely regular element, then a is regular, left regular, lateral regular and right regular.

***Proof* :** Suppose that a is completely regular.

Then there exist $x, y \in$ T such that $axaya = a$ and $axa = xaa = aax = aya = yaa = aay = axy = yxa = xay = yax$. Clearly A is regular.

Now $a = axaya = axaay = aaaxy = a^3xy$. Therefore a is left regular.

Also $a = axaya = xaaya = xaaay = xa^3y$. Therefore a is lateral regular.

and $a = axaya = xaaya = xyaaa = xya^3$. Therefore a is right regular.

THEOREM 2.77 : A cancellative left regular ternary semiring is commutative.

***Proof* :** Let T be a cancellative left regular ternary semiring. Let $a, b, c \in$ T

$\Rightarrow (abc)^3 = abcabcabc \Rightarrow a^3b^3c^3 = abcabcabc \Rightarrow a^2b^3c^2 = bcabcab \Rightarrow a^2b^2c^2 = bcacab$

$\Rightarrow (abc)(abc) = (bca)(cab) \Rightarrow abc = bca = cab$. Hence T is commutative. Therefore a cancellative left regular ternary semiring is commutative.

The following theorems 2.78, 2.79 and 2.80 are due to V. R. Daddi and Y. S. Pawar [3].

THEOREM 2.78 : Let T is a completely regular ternary semiring iff for any $a \in$ T, there exist $x \in$ T such that the following conditions are satisfied.

i) $a = a + x + a$ **ii)** $[(a + x)aa] = a + x$

THEOREM 2.79 : Let T is a completely regular ternary semiring iff for any $a \in$ T, there exist $x \in$ T such that the following conditions are satisfied.

i) $a = a + x + a$ **ii)** $[a(a + x)a] = a + x$

THEOREM 2.80 : Let T is a completely regular ternary semiring iff for any $a \in$ T, there exist $x \in$ T such that the following conditions are satisfied.

i) $a = a + x + a$
ii) $[aa(a + x)] = a + x$
iii) $[a(a + x)a] = a + x$
iv) $[(a + x)aa] = a + x$
v) $a + [(a + x)aa] = a$
vi) $a + [a(a + x)a] = a$
vii) $a + [aa(a + x)] = a$
viii) $[aa(a + x)] = [a(a + x)a] = [(a + x)aa]$.

DEFINITION 2.81: An element a of a ternary semiring T is said to be ***g-regular*** if there exist $x, y \in$ T such that $x = xayax$ or $y = yaxay$.

EXAMPLE 2.82 : Every cancellative regular ternary semiring is g-regular.

THEOREM 2.83 : A cancellative ternary semiring is regular if and only if it is *g*-regular.

***Proof* :** Let T be a cancellative ternary semiring. Assume that T is a regular ternary semiring. For any $a \in$ T there exist $x, y \in$ T such that $a = axaya \Rightarrow axa = axayaxa$. Since T is cancellative, $a(x)a = a(xayax)a \Rightarrow x = xayax \Rightarrow a$ is g-regular for every $a \in$ T. Therefore T is g-regular ternary semiring.

Conversely, Assume that T is g-regular ternary semiring. For any $a \in$ T there exist $x, y \in$ T such that $x = xayax \Rightarrow xax = xayaxax$. Since T is cancellative, $x(a)x = x(ayaxa)x \Rightarrow a = ayaxa \Rightarrow a$ is regular, for all a in T. Therefore, T is a regular ternary semiring.

In the following we are introducing the notion of mid unit of a ternary semiring.

DEFINITION 2.84 : An element a of a ternary semiring T is said to be a ***mid-unit*** provided $xayaz = xyz$ for all $x, y, z \in$ T.

DEFINITION 2.85 :. An element a of a ternary semiring T is said to be ***invertible*** in T if there exists an element b in T (called the ***ternary semiring-inverse*** *of* a) such that $abt = bat = tab = tba = t$ for all $t \in$ T.

DEFINITION 2.86 :. A ternary semiring (ring) T with $|S| \geq 2$ is said to be a ***ternary division semiring* (*ring*,** resp.) if every non-zero element of T is invertible.

THEOREM 2.87 : Every ternary division semiring is regular ternary semiring.

***Proof* :** From the definition 2.81, it follows that a bi-unital element of T is invertible and any invertible element is regular. Hence every ternary division semiring is a regular ternary semiring.

DEFINITION 2.88 : A commutative ternary division semiring (ring) is said to be a ***ternary semifield* (*field*,** resp.), i.e. a commutative ternary semiring (ring) T with $|T| \geq 2$, is a ternary semifield (field) if for every non-zero element a of T, there exists an element b in T such that $abx = x$ for all $x \in$ T.

NOTE 2.89 : A ternary semifield T has always an identity.

EXAMPLE 2.90 : Denote by R_0^-, Q_0^- and Z_0^- the sets of all non-positive real numbers, non-positive rational numbers and non-positive integers, respectively. Then R_0^- and Q_0^- form ternary semifields with usual binary addition and ternary multiplication and Z_0^- forms only a ternary semi-integral domain but not a ternary semifield.

IV. CONCLUSION

In this paper mainly we studied about some special elements in ternary semirings and some special type of ternary semirings.

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